

## NEW VACUUM OF BETHE ANSATZ SOLUTIONS IN THIRRING MODEL

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We find a new vacuum of the Bethe ansatz solutions in the massless Thirring model. This vacuum breaks the chiral symmetry and has the lower energy than the well-known symmetric vacuum energy. Further, we evaluate the energy spectrum of the one particle-one hole ( $1p-1h$ ) states, and find that it has a finite gap. The analytical expressions for the true vacuum as well as for the lowest  $1p-1h$  excited state are also found. Further, we examine the bosonization of the massless Thirring model and prove that the well-known procedure of bosonization of the massless Thirring model is incomplete because of the lack of the zero mode in the boson field.

KEYWORDS: Bethe ansatz, symmetry broken vacuum, zero mode

## 1. Introduction

Symmetries and their breaking have been one of the most important subjects in quantum field theory. Since the vacuum can break the symmetry of the field theory model, one learns the structure of the vacuum and its dynamics of the model through the symmetry breaking phenomena.<sup>1-3</sup>

In two dimensions, however, the symmetry breaking in the field theory is considered to be different from the four dimensional field theory models. In particular, Coleman<sup>4</sup> presented the proof that the two dimensional field theory models cannot spontaneously break the symmetry even though the vacuum state may prefer the symmetry broken state. However, his proof of the nonexistence of the spontaneous symmetry breaking in two dimensions is essentially based on the Goldstone theorem. The Goldstone theorem<sup>1,2</sup> states that the spontaneous symmetry breaking should accompany a massless boson when the vacuum prefers the broken symmetric state. However, the massless boson cannot exist in two dimensions since it cannot propagate due to the infra-red singularity of the propagator. Since this non-existence of the massless boson should hold rigorously, it naturally means that the spontaneous symmetry breaking should not occur in two dimensions as long as the Goldstone theorem is right.

Coleman's theorem looks reasonable, and indeed until recently it has been believed to hold true for fermion field theory models as well.

However, the recent work on the massless Thirring model shows that the chiral symmetry of the massless Thirring model is spontaneously broken by the Bogoliubov vacuum state.<sup>5-8</sup> There, the energy of the new vacuum is lower than that of the free vacuum state, and it indeed violates the chiral symmetry. It should be noted that, in this analysis, there appears no massless boson, and therefore it does not contradict the non-existence of the massless boson in two dimensions.

This claim, however, does not seem to be accepted yet in general since people may believe that the Bogoliubov

transformation does not have to be exact, and therefore there might be some excuse for the symmetry breaking phenomena that occurred in the Thirring model.

In this paper, we present a new discovery of the symmetry broken vacuum of the Bethe ansatz solution in the Thirring model, and show that the energy of the new vacuum state is indeed lower than that of the symmetric vacuum state even though the symmetric vacuum was considered to be the lowest state in the Thirring model. The new vacuum state breaks the chiral symmetry, and becomes a massive fermion field theory model.

Further, we evaluate the energy spectrum of the one particle-one hole states, and show that the excitation spectrum has indeed a finite gap. This gap energy turns out to be consistent with the effective fermion mass deduced from the momentum distribution of the negative energy particles in the new vacuum state. This confirms the consistency of the calculation of the Bethe ansatz solutions in the Thirring model.

After carrying out the numerical calculations, we get to know that the energies of the vacuum as well as the lowest one particle-one hole state can be expressed analytically. This is quite nice since we know clearly which of the vacuum state is the lowest. Also, in the thermodynamic limit, the lowest one particle-one hole state can be reduced to the effective fermion mass  $M_N$  which is described in terms of the cutoff  $\Lambda$ .

It turns out that there is no massive boson in the Bethe ansatz solutions, contrary to the prediction of the Bogoliubov transformation method.<sup>7,9</sup> However, qualitative properties of the symmetry breaking phenomena between the Bethe ansatz calculations and the Bogoliubov method agree with each other.

Even though the Bethe ansatz calculations confirm that there is no massless boson in the massless Thirring model, some people may claim that the massless Thirring model can be bosonized and is reduced to a massless boson hamiltonian. Here, we show that the well-known procedure of bosonization of the massless Thirring model is incomplete because the zero mode of the boson field cannot be defined and quantized. In other words, the zero mode of the field  $\Phi(0)$  identically vanishes in the massless Thirring model. This is in contrast to the Schwinger

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model in which one finds the zero mode of the field  $\Phi(0)$  by the gauge field  $A^1$ . Also, it is interesting to note that the massive Thirring model has the zero mode through the mass term, and this clearly indicates that the massless limit of the massive Thirring model is indeed a singular point with respect to the dynamics of the field theory.

Therefore, the massless Thirring model cannot be reduced to a free massless boson even though it has a similar mathematical structure to the massless boson. The spectrum of the massless Thirring model has a finite gap, and this is consistent with the fact that there should not be any physical massless boson in two dimensions. Even though the defect of the bosonization of the massless Thirring model is only one point of the boson field, that is, zero mode, it is interesting and surprising that *nature* knows it in advance.

This paper is organized as follows. In the next section, we discuss the Bethe ansatz solutions of the massless Thirring model. We obtain the analytic expressions for the vacuum energy as well as for the one particle one hole state excitation energy, and show that the new vacuum state breaks the chiral symmetry and has the lower energy than the symmetric vacuum state. In section 3, we present a critical review of the bosonization of the massless Thirring model, and show that the massless Thirring model cannot be bosonized properly and it has a spectrum with a finite gap. In section 4, we summarize what we clarify in this paper.

## 2. Thirring model and Bethe ansatz solutions

The massless Thirring model is a 1+1 dimensional field theory with current current interactions.<sup>10</sup> Its hamiltonian can be written as

$$H = \int dx \left\{ -i \left( \psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2 \right) + 2g \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right\}. \quad (2.1)$$

The hamiltonian eq.(2.1) can be diagonalized by the Bethe ansatz wave function for  $N$  particles<sup>10, 11, 14</sup>

$$\begin{aligned} |k_1, \dots, k_N\rangle &= \int dx_1 \dots dx_{N_1} dy_1 \dots dy_{N_2} \\ &\times \prod_{i=1}^{N_1} \exp(ik_i x_i) \prod_{j=1}^{N_2} \exp(ik_{N_1+j} y_j) \\ &\times \prod_{i,j} (1 + \lambda \theta(x_i - y_j)) \prod_{i=1}^{N_1} \psi_1^\dagger(x_i) \prod_{j=1}^{N_2} \psi_2^\dagger(y_j) |0\rangle, \end{aligned} \quad (2.2)$$

with  $N_1 + N_2 = N$ .  $\theta(x)$  denotes the step function.  $k_i$  represents the momentum of the  $i$ -th particle.  $\lambda$  is determined to be<sup>11</sup>

$$\lambda = -\frac{g}{2} S_{ij} \quad (2.3)$$

where  $S_{ij}$  is defined as

$$S_{ij} = \frac{k_i E_j - k_j E_i}{k_i k_j - E_i E_j - \epsilon^2}. \quad (2.4)$$

Here,  $\epsilon$  denotes the infra-red regulator which should be infinitesimally small. For the infra-red regulator  $\epsilon$ , it is important to note that the physical observables like momentum  $k_i$  do not depend on the regulator  $\epsilon$ . The derivation of eq.(2.3) is given in Appendix.

In this case, the eigenvalue equation becomes

$$H |k_1, \dots, k_N\rangle = \sum_{i=1}^N E_i |k_1, \dots, k_N\rangle. \quad (2.5)$$

From the periodic boundary condition (PBC), one obtains the following PBC equations,

$$k_i = \frac{2\pi n_i}{L} + \frac{2}{L} \sum_{j \neq i}^N \tan^{-1} \left( \frac{g}{2} S_{ij} \right) \quad (2.6)$$

where  $n_i$ 's are integer, and runs as  $n_i = 0, \pm 1, \pm 2, \dots, N_0$  where

$$N_0 = \frac{1}{2}(N-1).$$

### 2.1 Vacuum state

First, we want to make a vacuum. We write the PBC equations for the vacuum which is filled with negative energy particles<sup>12, 13</sup>

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L} \sum_{\substack{i \neq j \\ k_i \neq k_j}}^N \tan^{-1} \left( \frac{g}{2} \frac{k_i |k_j| - k_j |k_i|}{k_i k_j - |k_i| |k_j| - \epsilon^2} \right). \quad (2.7)$$

Although, the expression of  $S_{ij}$  is different from that of Odaka and Tokitake,<sup>14</sup> Andrei and Lowenstein,<sup>15</sup> it produces the same values of the Bethe ansatz solutions of the symmetric vacuum state. Here, we first fix the maximum momentum of the negative energy particles, and denote it by the cut off momentum  $\Lambda$ . Next, we take the specific value of  $N$ , and this leads to the determination of  $L$

$$L = \frac{2\pi N_0}{\Lambda}. \quad (2.8)$$

If we solve eq.(2.7), then we can determine the vacuum state, and the vacuum energy  $E_v$  can be written as

$$E_v = - \sum_{i=1}^N |k_i|. \quad (2.9)$$

It should be noted that physical observables are obtained by taking the thermodynamic limit where we let  $L \rightarrow \infty$  and  $N \rightarrow \infty$ , keeping  $\Lambda$  finite. If there is other scale like the mass, then one should take the  $\Lambda$  which is sufficiently large compared to the other scale. However, there is no other scale in the massless Thirring model or four dimensional QCD with massless fermions, and therefore all the physical observables are measured by the  $\Lambda$ . Here, we can take all the necessary steps, if required, since all the physical quantities are given analytically. In fact, as we see below, the excitation energy and the effective fermion mass are expressed in terms of the  $\Lambda$  in the thermodynamic limit.

## 2.2 Symmetric vacuum state

The solution of eq.(2.7) has been known and is written as<sup>10,14</sup>

$$k_1 = 0 \quad (2.10a)$$

for  $n_1 = 0$ ,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.10b)$$

for  $n_i = 1, 2, \dots, N_0$ ,

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.10c)$$

for  $n_i = -1, -2, \dots, -N_0$ . This gives a symmetric vacuum state, and was considered to be the lowest state.

The vacuum energy  $E_v^{\text{sym}}$  can be written as

$$E_v^{\text{sym}} = -\Lambda \left\{ N_0 + 1 + \frac{2N_0}{\pi} \tan^{-1} \left( \frac{g}{2} \right) \right\}. \quad (2.11)$$

## 2.3 True vacuum state

It is surprising that eq.(2.7) has a completely different solution from the above analytical solutions. By the numerical calculation of eq.(2.7), we first find the new vacuum state. After that, we get to know that the solutions can be analytically written like the symmetric case,

$$k_1 = \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.12a)$$

for  $n_1 = 0$ ,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.12b)$$

for  $n_i = 1, 2, \dots, N_0$ ,

$$k_i = \frac{2\pi n_i}{L} - \frac{2(N_0 + 1)}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.12c)$$

for  $n_i = -1, -2, \dots, -N_0$ . The new vacuum has no  $k_i = 0$  solution, and breaks the left-right symmetry. Instead, all of the momenta of the negative energy particles become finite.

The energy  $E_v^{\text{true}}$  of the true vacuum state can be written as

$$E_v^{\text{true}} = -\Lambda \left\{ N_0 + 1 + \frac{2(N_0 + 1)}{\pi} \tan^{-1} \left( \frac{g}{2} \right) \right\}. \quad (2.13)$$

From the distributions of the negative energy particles, one sees that this solution breaks the chiral symmetry. This situation can be easily seen from the analytical solutions since the absolute value of the momentum of the negative energy particles is higher than  $\frac{\Lambda}{\pi} \tan^{-1} \left( \frac{g}{2} \right)$ . Therefore, we can define the effective fermion mass  $M_N$  by

$$M_N = \frac{\Lambda}{\pi} \tan^{-1} \left( \frac{g}{2} \right). \quad (2.14)$$

In Table I, we show the calculated results of the new vacuum as well as the symmetric vacuum energies as the function of the particle number  $N$ . Here, we present the case with the coupling constant of  $g = \pi$ .

Table I. We show the calculated results of the vacuum energy of Bethe ansatz solutions at  $g = \pi$  with the particle number  $N = 401$  and  $N = 1601$ .  $\mathcal{E}_v^{\text{sym}}$  and  $\mathcal{E}_v^{\text{true}}$  denote the symmetric vacuum and the true vacuum energies, respectively. We also show the effective fermion mass  $\mathcal{M}_N$  deduced from the vacuum momentum distributions. All the energies are measured in units of  $\Lambda$ , namely,  $\mathcal{E} \equiv E/\Lambda$  and  $\mathcal{M}_N \equiv M_N/\Lambda$ .

$N$	$\mathcal{E}_v^{\text{sym}}$	$\mathcal{E}_v^{\text{true}}$	$\mathcal{M}_N$
401	-328.819	-329.458	0.320
1601	-1312.274	-1312.913	0.320

## 2.4 1p-1h state

Next, we evaluate one particle-one hole (1p-1h) states. There, we take out one negative energy particle ( $i_0$ -th particle) and put it into a positive energy state. In this case, the PBC equations become

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L} \tan^{-1} \left( \frac{g}{2} \frac{k_i |k_{i_0}| + k_{i_0} |k_i|}{k_i k_{i_0} + |k_i| |k_{i_0}| + \epsilon^2} \right) - \frac{2}{L} \sum_{\substack{j \neq i, i_0 \\ k_j \neq k_i, k_{i_0}}}^N \tan^{-1} \left( \frac{g}{2} \frac{k_i |k_j| - k_j |k_i|}{k_i k_j - |k_i| |k_j| - \epsilon^2} \right) \quad (2.15a)$$

for  $i \neq i_0$ .

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2}{L} \sum_{\substack{j \neq i_0 \\ k_j \neq -k_{i_0}}}^N \tan^{-1} \left( \frac{g}{2} \frac{k_{i_0} |k_j| + k_j |k_{i_0}|}{k_{i_0} k_j + |k_{i_0}| |k_j| + \epsilon^2} \right) \quad (2.15b)$$

for  $i = i_0$ . In this case, the energy of the one particle-one hole states  $E_{(i_0)}^{1p1h}$  is given as,

$$E_{(i_0)}^{1p1h} = |k_{i_0}| - \sum_{\substack{i=1 \\ i \neq i_0}}^N |k_i|. \quad (2.16)$$

It turns out that the solutions of eqs.(2.15) can be found at the specific value of  $n_{i_0}$  and then from this  $n_{i_0}$  value on, we find continuous spectrum of the 1p-1h states.

Here, we show the analytical solution of eqs.(2.15) for the lowest 1p-1h state.

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.17a)$$

for  $n_{i_0}$ ,

$$k_i = \frac{2\pi n_i}{L} + \frac{2(N_0 + 1)}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.17b)$$

for  $n_i = 0, 1, 2, \dots, N_0$

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L} \tan^{-1} \left( \frac{g}{2} \right) \quad (2.17c)$$

for  $n_i = -1, -2, \dots, -N_0$ .  $n_{i_0}$  is given by

$$n_{i_0} = \left\lceil \frac{N_0}{\pi} \tan^{-1} \left( \frac{g}{2} \right) \right\rceil, \quad (2.18)$$

where  $[X]$  denotes the smallest integer value which is larger than  $X$ . In this case, we can express the lowest 1p-1h state energy analytically

$$E_0^{1p-1h} = -\Lambda \left\{ (N_0 + 1) - \frac{2n_{i_0}}{N_0} + \frac{2(N_0 + 1)}{\pi} \tan^{-1} \left( \frac{g}{\pi} \right) \right\}. \quad (2.19)$$

Table II. We show several lowest states of the calculated results of the  $1p-1h$  states energy  $\mathcal{E}$  of eqs.(2.19) at  $g = \pi$  with  $N = 1601$ . The gap energy  $\Delta\mathcal{E} \equiv \mathcal{E}^{(1p1h)} - \mathcal{E}_v$  is also shown. All the energies are measured in units of  $\Lambda$ .

	$\mathcal{E}$	$\Delta\mathcal{E}$
vacuum	-1312.913	
$1p-1h$ (0)	-1312.273	0.640
$1p-1h$ (1)	-1312.272	0.641
$1p-1h$ (2)	-1312.271	0.642
$1p-1h$ (3)	-1312.269	0.644
$1p-1h$ (4)	-1312.268	0.645

Therefore, the lowest excitation energy  $\Delta E_0^{1p-1h}$  with respect to the true vacuum state becomes

$$\Delta E_0^{1p-1h} \equiv E_0^{1p-1h} - E_v^{\text{true}} = \frac{2\Lambda}{N_0} n_{i_0}. \quad (2.20)$$

If we take the thermodynamic limit, that is,  $N \rightarrow \infty$  and  $L \rightarrow \infty$ , then eq.(2.18) can be reduced to

$$\Delta E_0^{1p-1h} = \frac{2\Lambda}{\pi} \tan^{-1} \left( \frac{g}{2} \right) = 2M_N. \quad (2.21)$$

In Table II, we show the lowest five states of the  $1p-1h$  energy by the numerical calculation. From this, we can determine the gap energy.

From this gap energy, we can obtain the effective fermion mass which is one half of the lowest gap energy. This can be easily given as

$$M_N = 0.320 \Lambda. \quad (2.22)$$

This is consistent with the effective fermion mass deduced from the negative energy distribution of the vacuum. This confirms the consistency of the present calculations.

### 2.5 Boson state

In this calculation, we do not find any boson state, contrary to the prediction of the Bogoliubov transformation method. Since the present calculation is exact, we believe that the Bogoliubov calculation overestimates the attraction between the particle hole states. The main difference between the Bethe solutions and the Bogoliubov vacuum arises from the dispersion relation of the negative energy particles. From the Bethe ansatz solutions, it is clear that one cannot make a simple free particle dispersion with the fermion mass term while the Bogoliubov method assumes the free fermion dispersion relation for the negative energy particles. This should generate slightly stronger attraction for the Bogoliubov vacuum state than for the Bethe ansatz solution.

However, as far as the symmetry breaking mechanism is concerned, the Bogoliubov transformation gives a sufficiently reliable description of the dynamics in the spontaneous symmetry breaking phenomena.

## 3. Bosonizations

Here, we briefly review the bosonization procedure in two dimensional field theory models. In particular, we discuss the Schwinger model and the massless and massive Thirring models and show that the massless Thirring

model cannot be bosonized properly due to the lack of the zero mode of the boson field.

### 3.1 Schwinger model

The best known model of the bosonization is the Schwinger model<sup>16</sup> which is the two dimensional QED with massless fermions. In the Schwinger model, one takes a Coulomb gauge, and in this case, the space part of the vector potential  $A^1$  depends on time and corresponds to the zero mode of the boson field.<sup>17</sup> If one defines the fermion current  $j_\mu = \bar{\psi}\gamma_\mu\psi$ , then the momentum representation  $\tilde{j}_\mu$  of the current is related to the boson field and its conjugate field as

$$\tilde{j}_0(p) = ip\sqrt{\frac{L}{\pi}}\Phi(p) \quad \text{for } p \neq 0 \quad (3.1a)$$

$$\tilde{j}_1(p) = \sqrt{\frac{L}{\pi}}\Pi(p) \quad \text{for } p \neq 0 \quad (3.1b)$$

where  $\Phi(p)$  and  $\Pi(p)$  denote the boson field and its conjugate field, respectively.  $L$  denotes the box length.

It is very important to note that  $\Pi(0)$  and  $\Phi(0)$  are not defined in eqs.(3.1). In the Schwinger model, they are related to the chiral charge and its time derivative as

$$\Pi(0) = \frac{\pi}{g\sqrt{L}}Q_5 \quad (3.2a)$$

$$\Phi(0) = \frac{\pi}{g\sqrt{L}}\dot{Q}_5 \quad (3.2b)$$

where  $\dot{Q}_5$  is described by the vector field  $A^1$  due to the anomaly equation

$$\dot{Q}_5 = \frac{gL}{\pi}A^1. \quad (3.3)$$

From these identification, one can write down the hamiltonian for the Schwinger model

$$H = \sum_p \left\{ \frac{1}{2}\Pi^\dagger(p)\Pi(p) + \frac{1}{2}p^2\Phi^\dagger(p)\Phi(p) + \frac{g^2}{2\pi}\Phi^\dagger(p)\Phi(p) \right\}. \quad (3.4)$$

This is just the free massive boson hamiltonian.

### 3.2 Massless Thirring model

It has been believed that the massless Thirring model can be bosonized<sup>18,19</sup> in the same way as above, and its hamiltonian is written

$$H = \frac{1}{2} \sum_{p \neq 0} \left\{ \left(1 - \frac{g}{2\pi}\right)\Pi^\dagger(p)\Pi(p) + \left(1 + \frac{g}{2\pi}\right)p^2\Phi^\dagger(p)\Phi(p) \right\}. \quad (3.5)$$

This looks plausible, but one knows at the same time that the  $p = 0$  part is not included. In fact, there is a serious problem in the definition of the boson field  $\Phi(0)$  and  $\Pi(0)$  at the zero momentum  $p = 0$ . From eqs.(3.1), it is clear that one cannot define the zero mode of the boson field. In the Schwinger model, one finds the  $\Phi(0)$  due to the anomaly equation. However, the Thirring model has no anomaly, and therefore the  $\Phi(0)$  identically vanishes. That is,

$$\Phi(0) = 0. \quad (3.6)$$

There is no way to find the corresponding zero mode of the boson field in the massless Thirring model since the axial vector current is always conserved.

Therefore, the hamiltonian of the massless Thirring model eq.(3.5) does not correspond to the massless boson. It is interesting to notice that the problem is closely related to the zero mode which exhibits the infra-red property of the hamiltonian. This is just consistent with the non-existence of the massless boson due to the infra-red singularity of the propagator in two dimensions.<sup>20</sup> Further, as discussed in the previous section, the Bethe ansatz solutions confirm the finite gap of the massless Thirring spectrum, and this rules out a possibility of any excuse of the massless boson in the massless Thirring model.

### 3.3 Massive Thirring model

It is well known that the massive Thirring model is equivalent to the sine-Gordon field theory.<sup>21</sup> The proof of the equivalence is based on the observation that the arbitrary number of the correlation functions between the two models agree with each other if some constants and the fields of the two models are properly identified between them. This indicates that the massive Thirring model must be well bosonized.

This is now quite clear since the axial vector current conservation is violated by the mass term,

$$\partial_\mu j_\mu^5 = 2im\bar{\psi}\gamma_5\psi \quad (3.7)$$

where  $j_\mu^5$  is defined as

$$j_\mu^5 = \bar{\psi}\gamma_5\gamma_\mu\psi. \quad (3.8)$$

It should be noted that the  $j_0^5$  is equal to  $j_1$  in two dimensions.

Therefore, one can always define the  $\dot{Q}_5$  by

$$\dot{Q}_5 = 2im \int \bar{\psi}\gamma_5\psi dx. \quad (3.9)$$

Therefore, one obtains the field  $\Phi(0)$  of the boson in terms of eqs.(3.2) and (3.9).

$$\Phi(0) = \frac{2im\pi}{g\sqrt{L}} \int \bar{\psi}\gamma_5\psi dx. \quad (3.10)$$

### 3.4 Physics of zero mode

What is the physics behind the hamiltonian without the zero mode? Here, we discuss the effect of the zero mode and the eigenvalues of the hamiltonian in a simplified way. The hamiltonian eq.(3.5) can be rewritten as

$$H = H_B - \frac{1}{2} \left(1 - \frac{g}{2\pi}\right) \Pi^\dagger(0)\Pi(0) \quad (3.11)$$

where the  $\Pi(0)$  field is introduced by hand, and the existence of the  $\Pi(0)$  and  $\Phi(0)$  fields is assumed. Here,  $H_B$  denotes the free boson hamiltonian and is written as

$$H_B = \frac{1}{2} \sum_p \left\{ \left(1 - \frac{g}{2\pi}\right) \Pi^\dagger(p)\Pi(p) + \left(1 + \frac{g}{2\pi}\right) p^2 \Phi^\dagger(p)\Phi(p) \right\}. \quad (3.12)$$

Now, we assume the following eigenstates for  $H_B$  and  $\Pi^\dagger(0)\Pi(0)$  by

$$H_B|p\rangle = E_p|p\rangle \quad (3.13a)$$

$$\Pi^\dagger(0)\Pi(0)|\Lambda\rangle = \Lambda|\Lambda\rangle \quad (3.13b)$$

where  $E_p = \frac{2\pi}{L}p$  with  $p = 0, 1, 2, \dots$ , and  $\Lambda$  is related to the box length  $L$  by  $\Lambda = \frac{c_0}{L}$  with  $c_0$  constant.

Eq. (3.13a) is just the normal eigenvalue equation for the massless boson and its spectrum. On the other hand, eq.(3.13b) is somewhat artificial since the state  $|\Lambda\rangle$  is introduced by hand. The zero mode state of the hamiltonian  $H_B$  should couple with the state  $|\Lambda\rangle$ , and therefore new states can be made by the superposition of the two states

$$|v\rangle = c_1|\Lambda\rangle + c_2|0\rangle \quad (3.14)$$

where  $c_1$  and  $c_2$  are constants. Further, we assume for simplicity that the overlapping integral between the  $|0\rangle$  and the  $|\Lambda\rangle$  states is small and is given by  $\epsilon$

$$\langle 0|\Lambda\rangle = \epsilon. \quad (3.15)$$

In this case, the energy eigenvalues  $\langle v|H|v\rangle$  of eq.(3.11) become at the order of  $O(\epsilon)$

$$E_\Lambda = \langle \Lambda|H_B|\Lambda\rangle - \frac{1}{2} \left(1 - \frac{g}{2\pi}\right) \Lambda \quad (3.16a)$$

$$E_0 = -\frac{1}{2} \left(1 - \frac{g}{2\pi}\right) \langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle. \quad (3.16b)$$

If we assume that the magnitude of the  $\langle \Lambda|H_B|\Lambda\rangle$  and  $\langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle$  should be appreciably smaller than the  $\Lambda$ ,

$$\langle \Lambda|H_B|\Lambda\rangle \ll \Lambda \quad (3.17a)$$

$$\langle 0|\Pi^\dagger(0)\Pi(0)|0\rangle \ll \Lambda \quad (3.17b)$$

then the spectrum of the hamiltonian eq.(3.11) has a finite gap, and the continuum states of the massless excitations start right above the gap. This is just the same as the spectrum obtained from the Bethe ansatz solutions discussed in the previous section.

## 4. Conclusions

It has been believed for a long time that the Bethe ansatz solution of the massless Thirring model has only a symmetric solution for the vacuum, and this symmetric vacuum state has been considered to be the real vacuum since it was thought to be the lowest energy state.

Here, we have presented a symmetry broken vacuum of the Bethe ansatz solutions in the Thirring model, and have shown that the true vacuum energy is indeed lower than the symmetric vacuum energy. This is quite surprising since the symmetry preserving state often gives the lowest energy state in quantum mechanics. However, in the field theory model, there is also the case in which the symmetry is spontaneously broken in the vacuum state,<sup>22</sup> and this is indeed what is realized and observed in the Thirring model.

In this new vacuum state, the chiral symmetry is broken, and therefore the momentum distribution of the negative energy state become a massive fermion theory. From the distribution of the vacuum momentum, we deduce the fermion mass.

We have also calculated the one particle-one hole excitation spectrum, and found that the spectrum has a

finite gap. From this gap energy, we can determine the fermion mass, and confirm that the fermion mass from the gap energy agrees with the one which is estimated from the vacuum momentum distribution.

Also, we have shown that the bosonization procedure of the massless Thirring model has a serious defect since there is no corresponding zero mode of the boson field and that the massless Thirring model therefore cannot be fully bosonized.

Since the massless Thirring model cannot be bosonized properly, there is no massless excitation spectrum in the model, and this is consistent with the Bethe ansatz solutions that the massless Thirring model has a finite gap and then the continuum spectrum starts right above the gap.

Also, we should stress that the bosonization of the massless Thirring model has a subtlety, and one must be very careful for treating it. If one makes a small approximation or a subtle mistake in calculating the spectrum of the hamiltonian, then one would easily obtain unphysical massless excitations from the massless Thirring model. We believe that the same care must be taken for the  $SU(N)$  Thirring model where some approximations like the  $1/N$  expansion are made and the massless boson is predicted.<sup>23</sup> When we discuss the large  $N$  expansion, there are serious problems related to the  $1/N$  approximation. The basic point is that they cannot take into account the subtlety of the dynamics. In particular, if one makes first the large  $N$  limit, then one loses some important interactions which contribute to the boson mass. As Gross and Neveu pointed out in their paper,<sup>26</sup> the massless boson does not exist if they were to calculate to the higher orders in  $1/N$ . The existence of massless boson will give rise to infrared infinities arising from virtual states. This means that the lowest order approximation in  $1/N$  is meaningless, and to investigate the infrared stability of the theory one has to work to all orders in  $1/N$ . This infra-red problems become particularly important when treating the bound state like boson mass.

It is clear by now that the present results are in contradiction with Coleman's theorem.<sup>4</sup> In this paper, we have presented counter examples against Coleman's theorem, and the exact solutions in two dimensional field theory should correspond to "experimental facts". Therefore, one should figure out the mathematical reason why Coleman's theorem is violated in fermion field theory model. In addition to the massless Thirring model, QCD with massless fermions in two dimensions spontaneously breaks the chiral symmetry with the axial vector current conservation. Therefore, the massless QCD<sub>2</sub> is also in contradiction with Coleman's theorem, but there is no massless boson,<sup>24</sup> and in this sense, it does not violate the theorem that there should not exist any massless boson in two dimensions. In reality, there is no example of fermion field theory models in which the symmetry of the vacuum state is *not* broken due to Coleman's theorem. However, the basic and mathematical problem with Coleman's theorem is still unsolved in this paper and is left for the reader. But we believe that the basic problem of the symmetry breaking business in two dimensions must come from the Goldstone theorem itself for the fermion

field theory.<sup>7</sup>

At this point, we should make a comment on the correspondence between the Thirring model and the Heisenberg  $XXZ$  model. It is believed that the two models are equivalent to each other. However, it is now clear that the spectrum of the Thirring model gives a finite gap while the Heisenberg  $XXZ$  model predicts always gapless spectrum. This means that, even though the two models are mathematically shown to be equivalent to each other, they are physically very different.<sup>25</sup>

What should be the main reason for the difference? If one makes the field theory into the lattice, then the lattice field theory loses some important continuous symmetry like Lorentz invariance or chiral symmetry. If the lost symmetry plays some important role for the spectrum of the model, then the lattice field theory becomes completely a different model from the continuous field theory model.<sup>27</sup> In general, the way of cutting the continuous space into a discrete one is not unique, and equal cutting of the space may not be sufficient for some of the field theory models.

The present work is concerned with a specific model in two dimensions, and quite different from four dimensional field theory models. However, the present work certainly raises a warning on the lattice version of the field theory since clearly there are important continuous symmetries in any of the field theory models in four dimensions, and if these symmetries may be lost in the lattice version, then it is quite probable that the lattice calculations may not be able to reproduce a physically important spectrum of the continuous field theory models.

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## Appendix: Derivation of phase shift function at massless limit

We present the derivation of eq.(2.4) from the Bethe ansatz equations of the massive Thirring model which are given by Bergknoff and Thacker.<sup>11</sup> According to the Bethe ansatz, the Hamiltonian can be digonalized when the phase shift function  $S_{ij}$  is written as

$$S_{ij} = \frac{\sin(\theta_{k_i} - \theta_{k_j})}{\sin(\theta_{k_i} + \theta_{k_j})} \quad (\text{A.1})$$

where

$$\tan 2\theta_{k_i} = \frac{m_0}{k_i} \quad (\text{A.2})$$

Therefore, we can rewrite as

$$\sin \theta_{k_i} = \sqrt{\frac{1 - \cos 2\theta_{k_i}}{2}} = \frac{\sqrt{E_i - k_i}}{\sqrt{2E_i}}, \quad (\text{A.3a})$$

$$\cos \theta_{k_i} = \sqrt{\frac{1 + \cos 2\theta_{k_i}}{2}} = \frac{\sqrt{E_i + k_i}}{\sqrt{2E_i}}, \quad (\text{A.3b})$$

where  $E_i = \sqrt{k_i^2 + m_0^2}$ . In this case,  $S_{ij}$  becomes

$$\begin{aligned} S_{ij} &= \frac{\sqrt{E_i - k_i}\sqrt{E_j + k_j} - \sqrt{E_i + k_i}\sqrt{E_j - k_j}}{\sqrt{E_i - k_i}\sqrt{E_j + k_j} + \sqrt{E_i + k_i}\sqrt{E_j - k_j}} \\ &= \frac{k_i E_j - k_j E_i}{k_i k_j - E_i E_j - m_0^2}. \end{aligned} \quad (\text{A.4})$$

For the massless limit  $m_0 \rightarrow 0$ ,  $E_i = \sqrt{k_i^2 + m_0^2} \rightarrow |k_i|$ . Therefore, we have the phase shift function  $S_{ij}$  of the *massless* Thirring model with the *regulator*  $\epsilon$  as

$$S_{ij} = \frac{k_i |k_j| - k_j |k_i|}{k_i k_j - |k_i| |k_j| - \epsilon^2}. \quad (\text{A.5})$$

Here, it should be important to note that the solutions of eq.(2.7) do *not* depend on the regulator  $\epsilon$ . Therefore, we can take the massless limit properly.